ON THE AVERAGE RATE OF RETURN IN A CONTINUOUS TIME STOCHASTIC MODEL[1]

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Abstract. In a discrete time stochastic model of a pension investment funds market Gajek and Kaluszka(2000a) have provided a definition of the average rate of return which satisfies a set of economic correctnes postulates. In this paper the average rate of return is defined for a continuous time stochastic model of the market. The prices of assets are modeled by the multidimensional geometrical Brownian motion. A martingale property of the average rate of return is proven.

1. Introduction

Consider a group of n investment or pension funds. In Gajek and Kaluszka (2000a) the following definition of the average return of the group of funds during a given time period $[s, t]$ is proposed

$$\bar{r}(s,t) = \exp\left(\sum_{i=1}^{n} \int_s^t \frac{k_i(u) w_i(u)}{\sum_{j=1}^{n} k_j(u) w_j(u)} \delta_i(u) du\right) - 1, \tag{1.1}$$

where $k_i(t)$ and $w_i(t)$ denote a number of all units possessed by the members of the i-th fund and a value of the i-th fund unit at time t, respectively, and $\delta_i(t)$ is the instantaneous rate of interest for the accounting unit in the i-th fund defined as

$$\delta_i(t) = \frac{d}{dt} \log w_i(t) \tag{1.2}$$

Here and subsequently, the logarithm is to the base e. This definition possesses some desirable properties contrary to other definitions (see Gajek and Kaluszka (2000a))

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In Section 2 we introduce a definition of the average rate of return in a continuous time stochastic model based on a discrete time model presented in Gajek and Kaluszka(2000b). We assume that prices of assets are modelled as correlated geometric Brownian motions. This allows us to derive a proper definition of the average rate of return. The definition is given by

$$\bar{r}(s,t) = \exp\left(\sum_{i=1}^{n}\int_s^t A_i^*(u)\frac{dw_i(u)}{w_i(u)} + \int_s^t d\log A(u) - \int_s^t \frac{dA(u)}{A(u)} + \right.$$

$$\left. + \sum_{i=1}^{n}\int_s^t A_i^{*2}(u)\frac{dk_i(u)}{k_i(u)} - \sum_{i=1}^{n}\int_s^t A_i^{*2}(u) d\log k_i(u)\right) - 1. \tag{1.3}$$

Here and subsequently,

$$A_i(t) = k_i(t)w_i(t), \quad A(t) = \sum_{i=1}^{n} A_i(t), \quad A_i^*(u) = \frac{A_i(u)}{A(u)}, \tag{1.4}$$

and the intergals in (1.3) are the Itô integrals.

The suitability of the formula is investigated.

2. A continuous-time stochastic model for funds dynamics

2.1. The model. We consider the following state-variables:

$u_{ij}(t)$ = number of units of the j-th asset possessed by the i-th fund at time t, $i = 1, \dots, n$, $j = 1, \dots, N$.

$w_i(t)$ = value of a participation unit of the i-th fund at time t,

$k_i(t)$ = number of units of the i-th fund at time t,

$D_i(t)$ = inflow of contribution income minus an outflow of benefit payments of the i-th fund over the time period $[0, t]$,

$D(t) = \sum_{i=1}^{n} D_i(t)$,

$K_{ij}(t)$ = number of units which have flowed from the i-th fund to j-th one between time 0 and time T, where $j \neq i$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $\mathbb{F} = \{\mathcal{F}_t\}$ be a filtration, i.e. each $\mathcal{F}_t$ is an σ-algebra of subsets of Ω with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for every $s < t$. Without loss of generality, we assume that $\mathcal{F}_0 = \{\Omega, \emptyset\}$. The investors' planning horizont is T, a fixed positive number.

Assume we have N assets and their prices c_i, $i = 1, \dots, N$, are governed by the following stochastic differential equations:

$$dc_i(t) = c_i(t)\mu_i(t)dt + c_i(t)\sum_{j=1}^{N}\sigma_{ij}(t)dB_j(t) \quad \text{for} \quad i = 1, \dots, N, \tag{2.1}$$

where $(B_1(t), \dots, B_N(t))$ is a standard N-dimensional Brownian motion (under the real-world probability $\mathbb{P}$), and $\sigma_{ij}(t)$ and $\mu_i(t)$ are progressively measurable processes on $[0, T]$ with bounded variation, called volatility processes and mean return rate processes, respectively. We assume the matrix $(\sigma_{ij})_{i,j=1}^{N}$ is non-singular and assume that $\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^T \sigma_{ij}^2(t)dt < \infty$. Moreover, let us assume that $\sum_{i=1}^{N}\int_0^T |\mu_i(t)|dt < \infty$. Here and subsequently, the symbol $X = Y$ (resp. $X < Y$) means that the random variables X, Y are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}(X = Y) = 1$ (resp. $\mathbb{P}(X < Y) = 1$). The model (2.1) of prices of assets is commonly used in mathematical finance (see e.g. Musiela and Rutkowski(1997), Karatzas and Shreve(1998) or Shiryaev(1999)). By Itô's formula

$$c_i(t) = c_i(0)\exp\left(\int_0^t \left[\mu_i(s) - \frac{1}{2}\sum_{k=1}^{N}\sigma_{ik}^2(s)\right]ds + \sum_{k=1}^{N}\int_0^t \sigma_{ik}(s)dB_k(s)\right),$$

for $i = 1, \dots, N$.

The process D_i will be modelled as follows

$$dD_i(t) = \alpha_i(t)dt + \beta_i(t)dB_{i+N}(t), \quad i = 1, \dots, N, \tag{2.2}$$

where α_i, β_i are progressively measurable processess, and $\{B_i(t) : \ i = 1, 2, \dots, 2N\}$ are independent standard Brownian motions. The class (2.2) contains geometric Brownian motion which is commonly used as a model of inflow-outflow process (cf. Koo(1998)).

We assume that all investments are infinitely divisible. There are no transaction costs or taxes, the assets pay no dividends and member does not pay for allocation of his/her wealth. The split of units is not allowed. In addition, suppose that $A_i(t) > 0$ between times 0 and T for each i. Assuming strict positivity we avoids various technical complications.

The dynamics of a group of funds is described by the following stochastic differential equations:

$$w_i(t)k_i(t) = u_{i1}(t)c_1(t) + \ldots + u_{iN}(t)c_N(t), \tag{2.3}$$

$$k_i(t)dw_i(t) = u_{i1}(t)dc_1(t) + \ldots + u_{iN}(t)dc_N(t), \tag{2.4}$$

$$w_i(t)dk_i(t) = -w_i(t)\sum_{j\neq i} dK_{ij}(t) + \sum_{j\neq i} w_j(t)dK_{ji}(t) + dD_i(t), \tag{2.5}$$

$$w_i(t)dk_i(t) = c_1(t)du_{i1}(t) + \ldots + c_N(t)du_{iN}(t), \tag{2.6}$$

where $i = 1, 2, \ldots, n$, $t \in [0, T]$, and processes c_i, D_i are defined by (2.1) and (2.2), respectively. The equations (2.3)-(2.6) are counterparts of equations (2.8) and (2.10)-(2.12) of Gajek and Kaluszka(2000b). The functions u_{ij}, K_{ij} play a role of control variables. We also assume that K_{ij} is continuous stochastic process adapted to $\mathbb{F}$ with bounded variation on any compact interval.

After adding equations (2.5) we get

$$\sum_{i=1}^{n} w_i(t)dk_i(t) = dD(t). \tag{2.7}$$

Remark. In Bacinello(2000) (see also Chamorro and de Villarreal(2000)), the value $w_i(t)$ of participation unit of the i-th fund is modelled as geometric Brownian motion. In our setting, each random variable $w_i(t)$ is a sum of lognormaly distributed random variables. It seems that it is not possible to derive a proper definition of the average rate of return under the assumption of Bacinello(2000).

Recall Itô's formula for $f(\xi)$ with $\xi = \xi(t)$ such that $d\xi = \alpha dt + \sum \beta_j dB_j$:

$$\begin{aligned} df(\xi) &= f'(\xi)\sum \beta_j dB_j + \left(f'(\xi)\alpha + \frac{1}{2}f''(\xi)\sum \beta_j^2\right)dt \\ &= f'(\xi)d\xi + \frac{1}{2}f''(\xi)\left(\sum \beta_j^2\right)dt, \end{aligned} \tag{2.8}$$

where $f \in C^2(\mathbb{R})$, and α, β_i are progressively measurable processess (see also the local Itô's formula, e.g. Kallenberg(1997)).

Moreover, if $\xi_i = \alpha dt + \sum_j \beta_{ij} dB_j$ for $i = 1, 2$, then

$$d(\xi_1\xi_2) = \xi_1 d\xi_2 + \xi_2 d\xi_1 + (\sum \beta_{1j}\beta_{2j})dt. \tag{2.9}$$

2.2. Definition of the average return. Our definition of the average rate of return of a group of funds at a time interval $[s,t]$, say $\bar{r}(s,t)$, in the stochastic continuous-time model, is as follows

$$\bar{r}(s,t) = \exp\left(\sum_{i=1}^{n}\int_s^t A_i^*(u)\frac{dw_i(u)}{w_i(u)} + \int_s^t d\log A(u) - \int_s^t \frac{dA(u)}{A(u)} + \right.$$
$$\left. + \sum_{i=1}^{n}\int_s^t A_i^{*2}(u)\frac{dk_i(u)}{k_i(u)} - \sum_{i=1}^{n}\int_s^t A_i^{*2}(u) d\log k_i(u)\right) - 1. \tag{2.10}$$

Recall that $A_i(t)$ denotes the wealth of the i-th fund at time t, $A(t)$ means the global wealth of funds at time t, and $A_i^*(t)$ denotes the percentage of a relative value of assets of the i-th fund. The formula (2.10) is a generalization of (1.1) to the case of stochastic prices and random inflow-outflow process modelled by (2.1) and (2.2), respectively. In fact, if $\beta_i \equiv 0$, i.e. $dD_i(t) = \alpha_i(t)dt$, then

$$\bar{r}(s,t) = \exp\left(\sum_{i=1}^{n}\int_s^t A_i^*(u)\frac{dw_i(u)}{w_i(u)} + \int_s^t d\log A(u) - \int_s^t \frac{dA(u)}{A(u)}\right) - 1. \tag{2.11}$$

Moreover, if $dc_i(t) = \mu_i(t)dt$ for each i, then

$$\bar{r}(s,t) = \exp\left(\sum_{i=1}^{n}\int_s^t A_i^*(u)\frac{dw_i(u)}{w_i(u)}\right) - 1.$$

Let us now provide an heuristic argument for definition (2.11) based on the definition of the average rate of return $\bar{r}_A$ in a discrete time model:

$$\bar{r}_A(s,t) = \prod_{u=s}^{t-1}\left(1 + \sum_{i=1}^{n} A_i^*(u) r_i(u,u+1)\right) - 1.$$

proposed by Gajek and Kaluszka(2000b). By formula (2.15) of Gajek et al. (2000b)

$$\bar{r}_A(s,t) = \prod_{u=s}^{t-1}\frac{A(u+1) - d(u+1)}{A(u)} - 1$$
$$= \frac{A(t)}{A(s)}\exp\left(\sum_{u=s}^{t-1}\log(1 - \frac{D(u+1) - D(u)}{A(u)})\right) - 1$$
$$\approx \frac{A(t)}{A(s)}\exp\left(-\sum_{u=s}^{t-1}\frac{D(u+1) - D(u)}{A(u)}\right) - 1$$
$$\approx \frac{A(t)}{A(s)}\exp\left(-\int_s^t \frac{dD(u)}{A(u)}\right) - 1, \tag{2.12}$$

since $\ln(1-x) \approx -x$ if x is close to 0. The latter approximation in (2.12) follows from the definition of Itô's integral. Combining (2.3) and (2.6) we get

$$dD(u) = \sum_{i=1}^{n} w_i(u) dk_i(u) = dA(u) - \sum_{i=1}^{n} k_i(u) dw_i(u). \tag{2.13}$$

From (2.12) and (2.13) it follows

$$\begin{aligned}\bar{r}_A(s,t) &\approx \frac{A(t)}{A(s)} \exp\left(\sum_{i=1}^{n} \int_s^t k_i(u) \frac{dw_i(u)}{A(u)} - \int_s^t \frac{dA(u)}{A(u)}\right) - 1 \\ &= \frac{A(t)}{A(s)} \exp\left(\sum_{i=1}^{n} \int_s^t A_i^*(u) \frac{dw_i(u)}{w_i(u)} - \int_s^t \frac{dA(u)}{A(u)}\right) - 1.\end{aligned}$$

Since $A(t)$ is a strictly positive process,

$$\bar{r}_A(s,t) = \exp\left(\sum_{i=1}^{n} \int_s^t A_i^*(u) \frac{dw_i(u)}{w_i(u)} + \int_s^t d\log A(u) - \int_s^t \frac{dA(u)}{A(u)}\right) - 1 \tag{2.14}$$

(by the local Itô's formula), we obtain

$$\bar{r}_A(s,t) \approx \bar{r}(s,t)$$

as claimed.

2.3. Properties of the average rate of return (2.10)

Theorem 1. Suppose $\sigma_{ij}(\cdot)$ are bounded in t and ω. If $\{c_i(t), t \geq 0\}$ is an $\mathbb{F}$-martingale for every i, then $\{\bar{r}(0,t), t \geq 0\}$ is also an $\mathbb{F}$-martingale.

Proof. To simplify notation we cancel the dependence of time of considered processes. Recall that K_{ij} is adapted and continuous processes with bounded variation on any compact interval for each i, j. First observe that by (2.1), (2.2), (2.4) and (2.5) we have

$$dw_i = \frac{1}{k_i}\left(\sum_{j=1}^{N} c_j \mu_j u_{ij}\right) dt + \frac{1}{k_i} \sum_{l=1}^{N} \left(\sum_{j=1}^{N} c_j \sigma_{jl} u_{ij}\right) dB_l, \tag{2.15}$$

$$dk_i = \left(-w_i \sum_{j \neq i} k_{ij} + \sum_{j \neq i} w_j k_{ji} + \alpha_i\right) \frac{dt}{w_i} + \frac{\beta_i}{w_i} dB_{i+N}, \quad i = 1, 2, \dots, N, \tag{2.16}$$

where $dK_{ij} = k_{ij}dt$. By (2.3), (2.7), (2.15), (2.16) and Itô's formula (see (2.9)) we get

$$dA = \sum_{i=1}^{n} w_i dk_i + \sum_{i=1}^{n} k_i dw_i + (0 \cdot B_1 + \ldots + 0 \cdot B_N + B_{N+1} \cdot 0 + \ldots + B_{2N} \cdot 0)dt$$

$$= dD + \sum_{i=1}^{n} k_i dw_i = (\sum_{i=1}^{N} \alpha_i + \sum_{j=1}^{N} c_j \mu_j \sum_{i=1}^{n} u_{ij})dt +$$

$$+ \sum_{l=1}^{N} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij}) dB_l + \sum_{l=1}^{N} \beta_l dB_{l+N}. \tag{2.17}$$

By local Itô's formula we obtain

$$d \log A = \frac{1}{A} dA - \frac{1}{2A^2} \left(\sum_{l=1}^{N} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij})^2 + \sum_{i=1}^{N} \beta_i^2 \right) dt. \tag{2.18}$$

By (2.1) and (2.4)

$$\sum_{i=1}^{n} k_i dw_i = (\sum_{j=1}^{N} c_j \mu_j \sum_{i=1}^{n} u_{ij})dt + \sum_{l=1}^{N} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij}) dB_l. \tag{2.19}$$

Combining (2.18) and (2.19) one can get

$$\sum_{i=1}^{n} A_i^* \frac{dw_i}{w_i} + d \log A - \frac{dA}{A} = \frac{1}{A} \sum_{l=1}^{N} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij}) dB_l + \tag{2.20}$$

$$+ \left(\frac{1}{A} (\sum_{j=1}^{N} c_j \mu_j \sum_{i=1}^{n} u_{ij}) - \frac{1}{2A^2} \sum_{l=1}^{N} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij})^2 - \frac{1}{2A^2} \sum_{i=1}^{N} \beta_i^2 \right) dt.$$

In view of (2.16) and Itô's formula,

$$d \log k_i = \frac{dk_i}{k_i} - \frac{1}{2k_i^2} \left(\frac{\beta_i}{w_i} \right)^2 dt. \tag{2.21}$$

Hence

$$\sum_{i=1}^{n} A_i^{*2} \frac{dk_i}{k_i} - \sum_{i=1}^{n} A_i^{*2} d \log k_i = \frac{1}{2A^2} \sum_{i=1}^{N} \beta_i^2 dt. \tag{2.22}$$

From (2.20) and (2.22) it follows

$$\sum_{i=1}^{n} A_i^* \frac{dw_i}{w_i} + d \log A - \frac{dA}{A} + \sum_{i=1}^{n} A_i^{*2} \frac{dk_i}{k_i} - \sum_{i=1}^{n} A_i^{*2} d \log k_i$$

$$= \sum_{l=1}^{N} \frac{1}{A} (\sum_{j=1}^{N} c_j \sigma_{jl} \sum_{i=1}^{n} u_{ij}) dB_l +$$

$$+\left(\frac{1}{A}\sum_{j=1}^{N}c_j\mu_j\sum_{i=1}^{n}u_{ij}-\frac{1}{2A^2}\sum_{l=1}^{N}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})^2\right)dt. \qquad (2.23)$$

By Itô's formula for $f(x)=\exp(x)-1$ (see (2.8)) and (2.23) we have

$$d\bar{r}(0,t)=(\bar{r}(0,t)+1)\frac{1}{A}\sum_{l=1}^{N}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})dB_l+$$

$$+(\bar{r}(0,t)+1)\left(\frac{1}{A}\sum_{j=1}^{N}c_j\mu_j\sum_{i=1}^{n}u_{ij}-\frac{1}{2A^2}\sum_{l=1}^{N}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})^2+\right.$$

$$\left.+\frac{1}{2}\sum_{l=1}^{N}\frac{1}{A^2}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})^2\right)dt.$$

Hence

$$d\bar{r}(0,t)=(\bar{r}(0,t)+1)\frac{1}{A}\sum_{l=1}^{N}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})dB_l+$$

$$+(\bar{r}(0,t)+1)\frac{1}{A}(\sum_{j=1}^{N}c_j\mu_j\sum_{i=1}^{n}u_{ij})dt.$$

Since $c_i(t)$ is an $\mathbb{F}$-martingale for each i, we have $\mu_i(t)\equiv 0$ for each i, and, in consequence,

$$d(\bar{r}(0,t)+1)=(\bar{r}(0,t)+1)\frac{1}{A}\sum_{l=1}^{N}(\sum_{j=1}^{N}c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij})dB_l.$$

By assumption of boundedness of σ_{ij} we have

$$\mathbb{E}\exp\left(\frac{1}{2}\sum_{l=1}^{N}\int_0^T\Big(\frac{1}{A(t)}\sum_{j=1}^{N}\sum_{i=1}^{n}c_j(t)\sigma_{jl}(t)u_{ij}(t)\Big)^2dt\right)<\infty,$$

because $A(t)=\sum_{j=1}^N\sum_{i=1}^n c_j(t)u_{ij}(t)$. Hence from the Novikov condition it follows that $\{\bar{r}(0,t):0\le t\le T\}$ is a martingale (see e.g. Karatzas and Shreve(1998), p. 21, or Shiryaev(1999)). $\Box$

Remark. If $c_i(\cdot)$ is a submartingale for each i, and if short-selling is prohibited, then $\sum_{j=1}^N c_j\mu_j\sum_{i=1}^n u_{ij}\ge 0$ for every t so $\{\hat{r}(0,t):0\le t\le T\}$ is a submartingale.

Next, we formulate a list of properties which any properly defined average rate of return should possess. Some of them can be found in Kellison(1991). The average return $\bar{r}$ meets the demands.

Put

$$R(t) = \sum_{i=1}^{n} \int_s^t A_i^*(u) \frac{dw_i(u)}{w_i(u)} + \int_s^t d\log A(u) - \int_s^t \frac{dA(u)}{A(u)} +$$

$$+ \sum_{i=1}^{n} \int_s^t A_i^{*2}(u) \frac{dk_i(u)}{k_i(u)} - \sum_{i=1}^{n} \int_s^t A_i^{*2}(u) d\log k_i(u).$$

Property 1. If the group consists only the i-th fund, then

$$\bar{r}(s,t) = \frac{w_i(t) - w_i(s)}{w_i(s)}$$

Proof. Observe that

$$dR = \frac{dw_i}{w_i} + d\log k_i + d\log w_i - \frac{d(w_i k_i)}{w_i k_i} +$$

$$+ \frac{dk_i}{k_i} - d\log k_i = d\log w_i, \tag{2.24}$$

because of (2.15)-(2.16) and Itô's formula (2.9). From (2.24) we get

$$\bar{r}(s,t) = \exp\left(\int_s^t d\log w_i\right) - 1 = (w_i(t) - w_i(s))/w_i(s),$$

and the proof is completed. □

Property 2. For every $s < u < t$ with probability one,

$$1 + \bar{r}(s,t) = (1 + \bar{r}(s,u))(1 + \bar{r}(u,t))$$

Proof. A direct consequence of the following property of Itô's integral: for every $a < c < b$

$$\int_a^b \xi dB = \int_a^c \xi dB + \int_c^b \xi dB, \quad \mathbb{P} - a.s. \Box$$

Property 3. If on a subset of a probability space the accounting units of all funds have the same values over $[s,t]$, i.e. $w_1(u) = w_2(u) = \ldots = w_n(u)$ for every u, $s \le u \le t$, then the following equality

$$\bar{r}(s,t) = \frac{w_1(t) - w_1(s)}{w_1(s)}.$$

holds on the same subset.

Proof. Put $w = w_1$ and put $k = \sum_{i=1}^{n} k_i$. Of course,

$$A_i = wk_i, \quad A = wk.$$

Hence

$$dR = \sum_{i=1}^{n} A_i^* \frac{dw}{w} + d\log(wk) - \frac{d(wk)}{wk} + \sum_{i=1}^{n} \frac{k_i dk_i}{k^2} - \sum_{i=1}^{n} \left(\frac{k_i}{k}\right)^2 d\log k_i$$

$$= d\log w + d\log k - \frac{dk}{k} + \sum_{i=1}^{n} \frac{k_i dk_i}{k^2} - \sum_{i=1}^{n} \left(\frac{k_i}{k}\right)^2 d\log k_i. \tag{2.25}$$

Since $w_1 = \ldots = w_n = w$, we get from (2.2) and (2.7)

$$dk = \frac{1}{w}\sum_{i=1}^{n} \alpha_i dt + \sum_{i=1}^{N} \frac{\beta_i}{w} dB_{i+N}.$$

By the local Itô formula

$$d\log k = \frac{dk}{k} - \frac{1}{2k^2}\sum_{i=1}^{n} \left(\frac{\beta_i}{w}\right)^2 dt. \tag{2.26}$$

From (2.21), we obtain

$$\sum_{i=1}^{n} \frac{k_i dk_i}{k^2} - \sum_{i=1}^{n} \left(\frac{k_i}{k}\right)^2 d\log k_i = \sum_{i=1}^{n} \left(\frac{k_i}{k}\right)^2 \frac{1}{2k_i^2} \left(\frac{\beta_i}{w}\right)^2 dt$$

$$= \frac{1}{2(kw)^2}\sum_{i=1}^{n} \beta_i^2 dt. \tag{2.27}$$

Combining (2.25)-(2.27) yields

$$dR = d\log w,$$

which completes the proof. □

Property 4. Assume that there are reals $\alpha_i > 0$ and a function $\phi : [s,t] \to [0,\infty)$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $k_i(u) = \alpha_i \phi(u)$ for all $u \in [s,t]$, $i = 1, \ldots, n$. Then

$$\bar{r}(s,t) = \frac{\sum_{i=1}^{n} \alpha_i r_i w_i(s)}{\sum_{i=1}^{n} \alpha_i w_i(s)},$$

where $r_i = (w_i(t) - w_i(s))/w_i(s)$. Moreover, if we assume that the number of units of every fund is constant over the time interval $[s,t]$, i.e. $\phi(u_1) = \phi(u_2)$ for $s \le u_1 < u_2 \le t$, then

$$\bar{r}(s,t) = \frac{A(t) - A(s)}{A(s)}.$$

Proof. By assumption,

$$\sum_{i=1}^{n} A_i^{*2}\frac{dw_i}{w_i} = \frac{d(\sum_{i=1}^n \alpha_i w_i)}{\sum_{i=1}^n \alpha_i w_i}. \tag{2.28}$$

We conclude from (2.15) that

$$d\sum_{i=1}^{n}\alpha_i w_i = \left(\sum_{i=1}^{n}\frac{\alpha_i}{k_i}\sum_{j=1}^{N} c_j\mu_j u_{ij}\right)dt + \sum_{l=1}^{N}\left(\sum_{j=1}^{N} c_j\sigma_{jl}\sum_{i=1}^{n}\frac{\alpha_i}{k_i}u_{ij}\right)dB_l. \tag{2.29}$$

By (2.29) and the Itô formula

$$d\log(\sum_{i=1}^{n}\alpha_i w_i) = \frac{d(\sum_{i=1}^n \alpha_i w_i)}{\sum_{i=1}^n \alpha_i w_i} - \frac{1}{2(\sum_{i=1}^n \alpha_i w_i)^2}\sum_{l=1}^{N}\left(\sum_{j=1}^{N} c_j\sigma_{jl}\sum_{i=1}^{n}\frac{1}{\phi}u_{ij}\right)^2 dt$$

$$= \frac{d(\sum_{i=1}^n \alpha_i w_i)}{\sum_{i=1}^n \alpha_i w_i} - \frac{1}{2A^2}\sum_{l=1}^{N}\left(\sum_{j=1}^{N} c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij}\right)^2 dt. \tag{2.30}$$

Combining (2.28) and (2.30) yields

$$\sum_{i=1}^{n} A_i^{*2}\frac{dw_i}{w_i} = d\log(\sum_{i=1}^{n}\alpha_i w_i) + \frac{1}{2A^2}\sum_{l=1}^{N}\left(\sum_{j=1}^{N} c_j\sigma_{jl}\sum_{i=1}^{n}u_{ij}\right)^2 dt. \tag{2.31}$$

From (2.18) and (2.31) we get

$$\sum_{i=1}^{n} A_i^{*2}\frac{dw_i}{w_i} = d\log(\sum_{i=1}^{n}\alpha_i w_i) + \frac{1}{A}dA - d\log A - \frac{1}{2A^2}\sum_{i=1}^{N}\beta_i^2 dt. \tag{2.32}$$

By (2.22) and (2.32), one can obtain

$$dR = d\log(\sum_{i=1}^{n}\alpha_i w_i).$$

Consequently,

$$\bar{r}(s,t) = \exp\left(\int_s^t d\log(\sum_{i=1}^{n}\alpha_i w_i(u))du\right) - 1$$

$$= \frac{\sum_{i=1}^n \alpha_i w_i(t) - \sum_{i=1}^n \alpha_i w_i(s)}{\sum_{i=1}^n \alpha_i w_i(s)} = \frac{\sum_{i=1}^n \alpha_i r_i w_i(s)}{\sum_{i=1}^n \alpha_i w_i(s)}.$$

Moreover, if $\phi(u) = c$ for each u with $c > 0$, then

$$\bar{r}(s,t) = \frac{\sum_{i=1}^n c\alpha_i w_i(t) - \sum_{i=1}^n c\alpha_i w_i(s)}{\sum_{i=1}^n c\alpha_i w_i(s)} = \frac{A(t) - A(s)}{A(s)}.$$

This completes the proof. $\square$

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